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IMPLICATIONS OF PROPERTIES CONCERNING COMPLEMENTATION IN FINITE LATTICES

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ABSTRACT. In Lattice Theory one considers lattices with different types of complementating operations, like *complemented lattices*, *ortho-lattices* or *orthomodular lattices*. These types can be defined by elementary properties of unary mappings (like being antitone or fulfilling de-Morgan-laws). In this work, twelve simple attributes which suffice to define the most important notions of complementations are investigated. A complete list of valid implications between them is presented, as well as a list of counterexamples for implications which are not valid.

1. MOTIVATION

In [DW] it turned out that for a better modal understanding of triadic contexts, the class of finite lattices with an involutorial semi-complementation should be investigated. In Lattice Theory there are many different notions of lattices with unary complementations, like "complemented lattices" or "ortho-lattices", and in this work we want to examine how involutorial semi-complementations are related to other kinds of complementations. This is done as follows: The most common notions concerning complemented lattices are broken up into simpler properties such that the different notions of being complemented can be composed with these properties. Between these properties, a sufficient list of valid implications between them is presented, that means every valid implication is a logical consequence of this list. On the other hand, a complete list of counterexamples for implications which are not valid is presented, too (with "implication" we denote formulas of first order predicate calculus of the form $\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \phi_{n+1}$). In particular, we get all dependencies between involutorial semi-complementations and other notions of complementations.

2. PROPERTIES

Let $\underline{L} = (L, \wedge, \vee, 0, 1, ^{\perp})$ be a finite lattice with a unary operation $^{\perp}$. The following properties shall be investigated:

1. $\underline{L}$ is modular
2. $\underline{L}$ is distributive
3. $x^{\perp} = 0 \Rightarrow x = 1$
4. $x^{\perp} = 1 \Rightarrow x = 0$
5. $x \vee x^{\perp} = 1$ (main property for a $\vee$-semi-complement)
6. $x \wedge x^{\perp} = 0$ (main property for a $\wedge$-semi-complement)
7. $x \leq y \Rightarrow x \vee (x^{\perp} \wedge y) = y$ (main property for orthomodularity)
8. $y \leq x \Rightarrow x \wedge (x^{\perp} \vee y) = y$ (main property for dual orthomodularity)
9. $x^{\perp\perp} = x$ ($^{\perp}$ is involutorial)
10. $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$ ($^{\perp}$ is antitone)
11. $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$ ($\vee$-de-Morgan)
12. $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$ ($\wedge$-de-Morgan)

With these properties, the following classic definitions can be performed:

1. $x^{\perp}$ is a $\underline{\vee\text{-semi-complement}}$ of x, when 4) and 5) hold.
2. $x^{\perp}$ is a $\underline{\wedge\text{-semi-complement}}$ of x, when 3) and 6) hold.
3. $x^{\perp}$ is a $\underline{\text{(full) complement}}$ of x, when 5) and 6) hold.
4. $\underline{\text{L}}$ is an $\underline{\text{ortholattice}}$, when $^{\perp}$ is a full complementation which fulfills 9) and 10). If 1) additionally holds, L is called $\underline{\text{modular ortholattice}}$.
5. $\underline{\text{L}}$ is an $\underline{\text{orthomodular lattice}}$, when L is an ortholattice and $^{\perp}$ additionally fulfills 7).
6. $\underline{\text{L}}$ is a $\underline{\text{boolean lattice}}$, when L is distributive and $^{\perp}$ is a full complementation.

3. Examples and Context

In figures 1–6, the Hasse-diagrams of all lattices $\underline{L} := (L, \wedge, \vee, 0, 1, ^{\perp})$ we need are listed.

In the first examples, the operation $^{\perp}$ is indicated by dotted lines having arrows at their ends. A double arrow between two elements x and y is to be read as $x^{\perp} = y$ and $y^{\perp} = x$.

In the further examples the elements of the lattice are labelled, and the operation $^{\perp}$ is given by a table. Again, a double arrow between two elements x and y is to be read as $x^{\perp} = y$ and $y^{\perp} = x$.

Which of the properties apply to which examples is encoded in the following formal context. Note that the dual of a lattice is denoted by the number of the lattice followed by a "d" (*dual* means that the order on the lattice is reversed, but the operation $^{\perp}$ is not changed).

In figure 8 one can see a nested diagram of the concept lattice of the context in figure 7. The lattice is derived by using *Formal Concept Analysis* (see [GW]). The elements of the lattice are the black filled points in the diagram. The diagram can be seen as the product of an inner and an outer Hasse-diagram, and one point lies below another iff this is the case in the inner

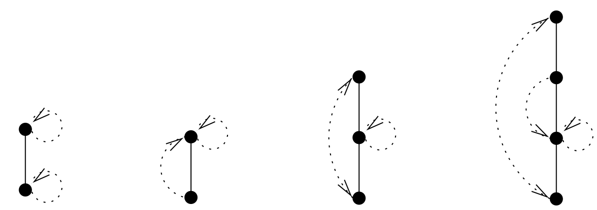

FIGURE 1. examples 1 to 4

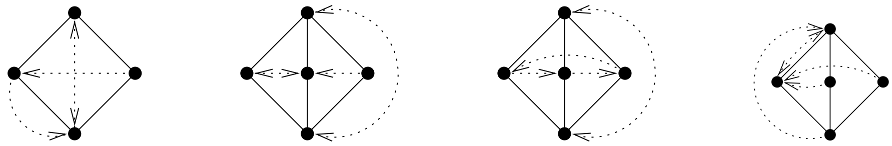

FIGURE 2. examples 5 to 8

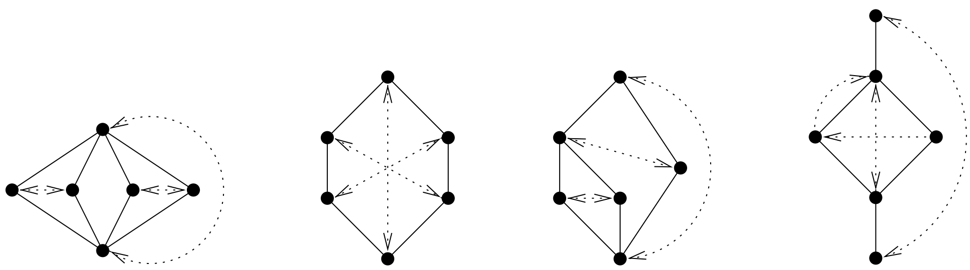

FIGURE 3. examples 9 to 11

diagram *and* in the outer diagram (for the enclosing big circles). For example, the element which is labelled with "03" lies below the element which is labelled with "01", but not below the element which is labelled with "02".

Further information on nested diagrams can be found in [GW].

4. A Basis for the Implications

The following propositions yield a basis for all implications which hold in the formal context, e.g. in the set of all examples. This has been verified by using the program ConImp of Peter Burmeister (see [Bu]). This program relies on a method of Formal Concept Analysis (see [GW]), namely the *attribute exploration.* ConImp gives for a given formal context a basis (the

Duquenne-Guiges-Basis) for all implications which are valid in the context. The implications listed in this section are the Duquenne-Guiges-Basis for the context given in figure 7.

Because these implications are all provable for finite lattices, any implication which holds in all examples holds in any finite lattice, or, vice versa: Any implication which does not hold in every finite lattice has a counterexamples among the examples listed here.

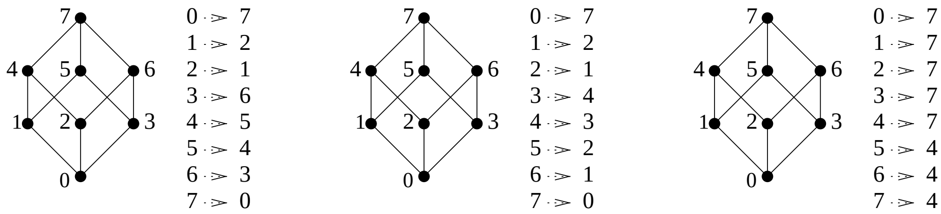

FIGURE 4. examples 13 to 15

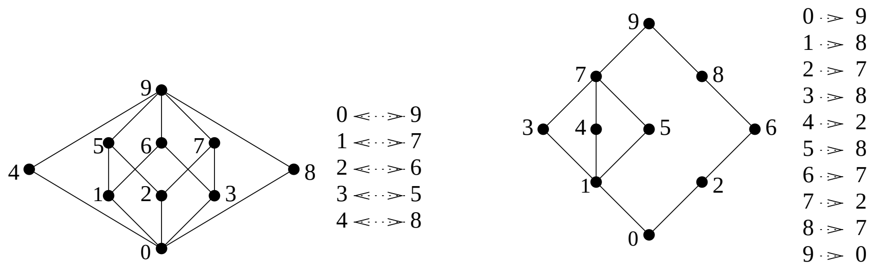

FIGURE 5. examples 16 to 17

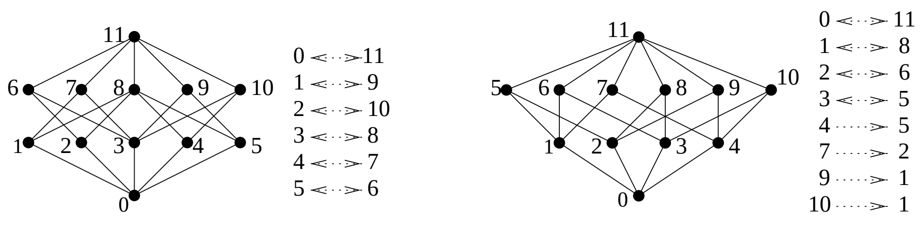

FIGURE 6. examples 18 to 19

	modular	distributiv	x'=0 => x=1	x'=1 => x=0	sup-semi-comp	inf-semi-comp	orthomodular	dual orthomodular	involutorial	antitone	sup-de-Morgan	inf-de-Morgan
1	×	×							×			
2	×	×	×		×		×			×	×	×
2d	×	×		×		×		×		×	×	×
3	×	×	×	×					×	×	×	×
4	×	×	×	×						×	×	×
5	×	×		×		×		×		×	×	
5d	×	×	×		×		×			×		×
6	×		×	×	×	×	×	×		×		
7	×		×	×	×	×	×	×		×	×	×
8	×		×		×		×			×	×	
8d	×			×		×		×		×		×
9	×		×	×	×	×	×	×	×	×	×	×
10			×	×	×	×			×	×	×	×
11			×	×		×			×			
11d			×	×	×				×			
12	×	×	×	×						×		×
12d	×	×	×	×						×	×	
13	×	×	×	×					×			
14	×	×	×	×		×		×				
14d	×	×	×	×	×		×					
15	×	×	×		×		×			×		
15d	×	×		×		×		×		×		
16			×	×	×	×	×	×	×	×	×	×
17			×	×	×	×				×		×
17d			×	×	×	×				×	×	
18	×		×	×	×	×	×	×	×			
19			×	×	×	×		×		×		
19d			×	×	×	×	×			×		

FIGURE 7. The context

Most of the propositions are trivial or well known, but they are listed for sake of completeness. Nevertheless, at least the last three theorems are non-trivial.

Of course, for every proposition the dual proposition holds, too. If a proposition is not selfdual, both forms are listed as formal implications using the numbers of the properties. But, for sake of clarity, only one form is written in terms and proven afterwards.

Proposition 1. *2)* $\Longrightarrow$ *1)*

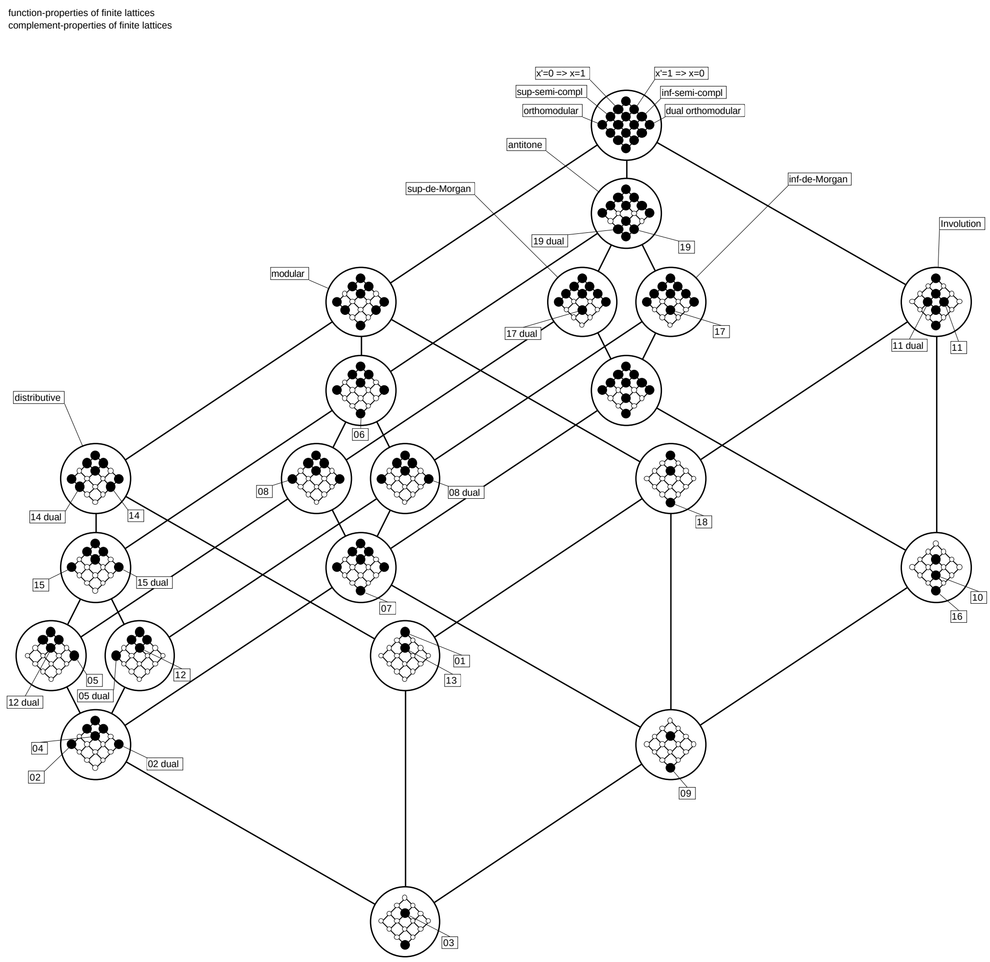

Figure 8. The concept lattice

If $\underline{L}$ is distributive, then $\underline{L}$ is modular.

Proposition 2. *5) $\Longrightarrow$3) and dually 6) $\Longrightarrow$4)*

If the main property of a $\vee$-semi-complement holds, then $x \neq 1 \Rightarrow x^{\perp} \neq 0$.

Proof. Let $x \neq 1$. From $x \vee x^{\perp} = 1$ we conclude $x^{\perp} \neq 0$. q.e.d.

Proposition 3. *7) $\Longrightarrow$5) and dually 8) $\Longrightarrow$6).*

If the main property of orthomodularity holds, then the main property of a $\vee$-semi-complement holds.

Proof. $x \leq 1 \Longrightarrow x \vee x^{\perp} = x \vee (x^{\perp} \wedge 1) \stackrel{7)}{=} 1$ q.e.d.

Proposition 4. *11) $\Longrightarrow$10) and dually, 12) $\Longrightarrow$10)*

If $\vee$-de-Morgan holds, then $^\perp$ is antitone.

Proof. Let $x \leq y$. We conclude $y^\perp = (x \vee y)^\perp = x^\perp \wedge y^\perp$, so $y^\perp \leq x^\perp$. q.e.d.

Proposition 5. *1), 5) $\Longrightarrow$7) and dually 1), 6) $\Longrightarrow$8)*

If $\underline{L}$ is modular and $^\perp$ fulfills the main property of a $\vee$-semi-complement, then the main property of orthomodularity holds.

Proof. Let $x \leq y$. We conclude $x \vee (x^\perp \wedge y) \overset{1)}{=} (x \vee x^\perp) \wedge y \overset{\vee 5)}{=} 1 \wedge y = y$ q.e.d.

Proposition 6. *9), 3) $\Longrightarrow$4) and dually 9), 4) $\Longrightarrow$3)*

If $^\perp$ is an involutorial operation which fulfills $x \neq 1 \Rightarrow x^\perp \neq 0$, then it fulfills $x \neq 0 \Rightarrow x^\perp \neq 1$, too.

Proof. From $0 = 0^{\perp\perp} = (0^\perp)^\perp$ and $x \neq 1 \Rightarrow x^\perp \neq 0$ we conclude $0^\perp = 1$, hence $1^\perp = 0$. Because $^\perp$ is bijective, we gain $x \neq 0 \Rightarrow x^\perp \neq 1$. q.e.d.

Proposition 7. *9), 10) $\Longrightarrow$11), 12), 3), 4)*

If $^\perp$ is an antitone involution, then both de-Morgan-laws, $x \neq 1 \Rightarrow x^\perp \neq 0$ and $x \neq 0 \Rightarrow x^\perp \neq 1$ hold.

Proof. An antitone involution is an anti-automorphism. In particular, both de-Morgan-laws hold, and we gain $1^\perp = 0$ as well as $0^\perp = 1$. Because $^\perp$ is bijective, we conclude $x \neq 1 \Rightarrow x^\perp \neq 0$ and $x \neq 0 \Rightarrow x^\perp \neq 1$. q.e.d.

Proposition 8. *4), 5), 10) $\Longrightarrow$6) and dually 3), 6), 10) $\Longrightarrow$5)*

If $^\perp$ is antitone and fulfills $x \neq 0 \Rightarrow x^\perp \neq 1$ and the main property of a $\vee$-semi-complement, then it fulfills the main property of a $\wedge$-semi-complement, too.

Proof. Because $^\perp$ is antitone, from $x \wedge x^\perp \leq x$ and $x \wedge x^\perp \leq x^\perp$ we conclude $(x \wedge x^\perp)^\perp \geq x^\perp$ and $(x \wedge x^\perp)^\perp \geq x^{\perp\perp}$. So we gain $(x \wedge x^\perp)^\perp \geq x^\perp \vee x^{\perp\perp}$. Because of the main property of a $\vee$-semi-complement we infer $x^\perp \vee x^{\perp\perp} = 1$, hence $(x \wedge x^\perp)^\perp = 1$. Due to $x \neq 0 \Rightarrow x^\perp \neq 1$ we ensure $(x \wedge x^\perp) = 0$. q.e.d.

It is well known that the finite, distributive and complemented lattices are, up to isomophism, exactly the finite powersets with the set-theoretical complement. In particular we gain

Proposition 9. *2), 5), 6) $\Longrightarrow$1), 3), 4), 7), 8), 9), 11), 12).*

If $\underline{L}$ is distributive and if $^\perp$ is a full complementation, then all remaining properties hold.

Proposition 10. *2), 5), 11) $\Longrightarrow$2)*

If $\underline{L}$ is distributive, if $^\perp$ fulfills the main property of a $\vee$-semi-complement and if the $\vee$-de-Morgan-law holds, then the $\wedge$-de-Morgan-law holds, too.

Proof. First, the $\wedge$-de-Morgan-law shall be shown for $\vee$-irreducible elements. So let $a, b \in L$ be $\vee$-irreducible. If a and b are comparable, we gain immediately $(a \wedge b)^\perp = a^\perp \vee b^\perp$ because $^\perp$ is antitone. So let a and b be incomparable. It holds $a = a \wedge (b \vee b^\perp) = (a \wedge b) \vee (a \wedge b^\perp)$. Because a and b are incomparable, we gain $a \wedge b < a$ and, because a is $\vee$-irreducible, $a = a \wedge b^\perp$, hence $a \leq b^\perp$. We gain even $a^\perp \vee b^\perp \geq a^\perp \vee a = 1$, so $a^\perp \vee b^\perp = 1$. On the other hand $(a \wedge b)^\perp \geq a^\perp \vee b^\perp$ holds because $^\perp$ is antitone, and so we conclude $(a \wedge b)^\perp = a^\perp \vee b^\perp = 1$. Therefore the $\wedge$-de-Morgan-law is proven for $\vee$-irreducible elements.

Now let x, y be arbitrary elements of L. Then x and y can be written as $x = \bigvee_{i=1,\dots,m} a_i$ and $y = \bigvee_{j=1,\dots,n} b_j$ with $\vee$-irreducible elements a_i, $i = 1, \dots, m$ and b_j, $j = 1, \dots, n$. Now we compute

$$\begin{aligned}
(x \wedge y)^\perp &\overset{\text{s.a.}}{=} \Big(\big(\bigvee_{i=1,\dots,m} a_i\big) \wedge \big(\bigvee_{j=1,\dots,n} b_j\big)\Big)^\perp \\
&\overset{2)}{=} \Big(\bigvee_{\substack{i=1,\dots,m \\ j=1,\dots,n}} (a_i \wedge b_j)\Big)^\perp \\
&\overset{11)}{=} \bigwedge_{\substack{i=1,\dots m \\ j=1,\dots,n}} (a_i \wedge b_j)^\perp \\
&\overset{\text{s.a.}}{=} \bigwedge_{\substack{i=1,\dots,m \\ j=1,\dots,n}} (a_i^\perp \vee b_j^\perp) \\
&\overset{2)}{=} \Big(\big(\bigwedge_{i=1,\dots,m} a_i^\perp\big) \vee \big(\bigwedge_{j=1,\dots,n} b_j^\perp\big)\Big) \\
&\overset{11)}{=} \Big(\big(\bigvee_{i=1,\dots,m} a_i\big)^\perp \vee \big(\bigvee_{j=1,\dots,n} b_j\big)^\perp\Big) \\
&\overset{\text{s.a.}}{=} x^\perp \vee y^\perp
\end{aligned}$$

q.e.d.

Before proving the next theorem, the following lemma has to be shown:

Lemma 1. *If $^\perp$ fulfills the main properties of orthomodularity and of a $\wedge$-semi-complement and if one de-Morgan-law holds, then $^\perp$ is bijective. The dual holds, too.*

Proof. Suppose there are elements x, y with $x \neq y$ and $x^\perp = y^\perp$. The proof is first done using the $\vee$-de-Morgan-law. We distinguish the following cases:

1. Let x and y be comparable. W.l.o.g. let $x < y$. We conclude

$$y \overset{7)}{=} x \vee (x^\perp \wedge y) \overset{\text{s.a.}}{=} x \vee (y^\perp \wedge y) \overset{6)}{=} x \vee 0 = x$$

2. Let x and y be incomparable. Define $z = x \vee y$. It follows $z > x$ and

$$z^\perp = (x \vee y)^\perp \overset{11)}{=} x^\perp \wedge y^\perp \overset{\text{s.a.}}{=} x^\perp \wedge x^\perp = x^\perp$$

So the second case can be reduced to the first one.

In the case of the $\wedge$-de-Morgan-law instead of the $\vee$-de-Morgan-law, simply define $z = x \wedge y$ (instead of $z = x \vee y$). The remaining arguments are analogous to the $\vee$-de-Morgan-case. q.e.d.

Now the next theorem can be stated:

Proposition 11. *7), 6), 11) $\Longrightarrow$8), 12) and dually 8), 5), 12) $\Longrightarrow$7), 11).*
7), 6), 12) $\Longrightarrow$8), 11) and dually 8), 5), 11) $\Longrightarrow$7), 12)

If $^\perp$ fulfills the main property of orthomodularity and a $\wedge$-semi-complement and if one de-Morgan-law holds, then the other de-Morgan-law and the main property of dual orthomodularity hold, too.

Proof. Due to theorem 4, $^\perp$ is antitone, and due to lemma 1, $^\perp$ is bijective. A bijective and antitone self-mapping ϕ of a finite lattice is an anti-automorphism. If an anti-automorphism ϕ is additionally compatible with $^\perp$ (that means $\phi(x^\perp) = \phi(x)^\perp$), both de-Morgan-laws hold, and with every proposition the dual proposition holds, too. For taking ϕ as $^\perp$, this presupposition trivially holds, so we gain immediately the conclusions of the stated theorem. q.e.d.

In proposition 11, we can additionally conclude from the given semi-complement-property to the dual semi-complement-property, too. This follows at once from theorem 3.

Proposition 12. *7), 9) $\Longrightarrow$8) and dually 8), 9) $\Longrightarrow$7)*

If $^\perp$ is involutorial and fulfills the main property of orthomodularity, then it fulfills the main property of dual orthomodularity, too.

Proof. For each $x \in L$ we define the following operation:

$$\phi_x : \begin{cases} \uparrow x & \rightarrow \quad \downarrow x^\perp \\ \quad y & \mapsto \quad x^\perp \wedge y \end{cases}$$

(where $\downarrow x := \{y \,|\, x \leq y\}$). First of all, it shall be shown that each ϕ_x is an order-isomorphism. Obviously, each ϕ_x is well-defined and isotone. On the other hand, for $y_1, y_2 \geq x$ we conclude $\phi_x(y_1) \leq \phi_x(y_2)$ and therefore:

$$y_1 \stackrel{7)}{=} x \vee (x^\perp \wedge y_1) = x \vee \phi_x(y_1) \leq x \vee \phi_x(y_2) = x \vee (x^\perp \wedge y_2) \stackrel{7)}{=} y_2$$

Hence every ϕ_x is even an order-embedding. In particular, it is injective, so we gain $|\uparrow x| \leq |\downarrow x^\perp|$ for each $x \in L$. This implies

$$|\leq| = \sum_{x\in L} |\uparrow x| \leq \sum_{x\in L} |\downarrow x^\perp| \stackrel{\perp\text{bij.}}{=} \sum_{x\in L} |\downarrow x| = |\leq|$$

So in the inequation, the "$\leq$" is in fact an "$=$" which yields $\sum_{x\in L} |\uparrow x| = \sum_{x\in L} |\downarrow x^\perp|$. Because we have $|\uparrow x| \leq |\downarrow x^\perp|$ for each summand, this in turn implies $|\uparrow x| = |\downarrow x^\perp|$ for each $x \in L$. So every ϕ_x is bijective and therefore an order-isomophism.

Now define:

$$\psi_x : \begin{cases} \downarrow x^{\perp} & \to & \uparrow x \\ y & \mapsto & x \vee y \end{cases}$$

For $y \geq x$ we conclude: $\psi_x(\phi_x(y)) = x \vee (x^{\perp} \wedge y) \overset{7)}{=} y$. Therefore the mapping $\psi_x \circ \phi_x$ is in fact the identity on $\uparrow x$, which implies that ψ_x is an ismorphism. For $y \in \downarrow x^{\perp}$ we obtain $\phi_x(\psi_x(y)) = y$, e.g.

$$y \leq x^{\perp} \Longrightarrow y = x^{\perp} \wedge (x \vee y)$$

Since $^{\perp}$ is involutorial, we can exchange x with $x^{\perp}$ (and vice versa), which finally yields

$$y \leq x \Longrightarrow y = x \wedge (x^{\perp} \vee y) \quad .$$ q.e.d.

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