Two Instances of Peirce's Reduction Thesis

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Abstract. A main goal of Formal Concept Analysis (FCA) from its very beginning has been the support of rational communication by formalizing and visualizing concepts. In the last years, this approach has been extended to traditional logic based on the doctrines of concepts, judgements and conclusions, leading to a framework called *Contextual Logic*. Much of the work on Contextual Logic has been inspired by the Existential Graphs invented by Charles S. Peirce at the end of the 19th century. While his graphical logic system is generally believed to be equivalent to first order logic, a proof in the strict mathematical sense cannot be given, as Peirce's description of Existential Graphs is vague and does not suit the requirements of contemporary mathematics.

In his book 'A Peircean Reduction Thesis: The Foundations of topological Logic', Robert Burch presents the results of his project to reconstruct in an algebraic precise manner Peirce's logic system. The resulting system is called Peircean Algebraic Logic (PAL). He also provides a proof of the Peircean Reduction Thesis which states that all relations can be constructed from ternary relations in PAL, but not from unary and binary relations alone.

Burch's proof relies on a major restriction on the allowed construction of graphs. Removing this restriction renders the proof much more complicated. In this paper, a new approach to represent an arbitrary graph by a relational normal form is introduced. This representation is then used to prove the thesis for infinite and two-element domains.

1 Introduction

From its very beginning, FCA was not only understood as an approach to restructure lattice theory (see [Wil82]) but also as a method to support rational communication among humans and as a concept-oriented knowledge representation. While FCA supports communication and argumentation on a concept level, an extended approach was needed to also support the representation of judgments and conclusions. This led to the development of contextual logic (see [DK03, Wil00]).

Work on contextual logic has been influenced by the Conceptual Graphs invented by John Sowa (see [Sow84, Sow92]). These graphs are in turn inspired by the Existential Graphs from Charles S. Peirce. In Peirce's opinion the main purpose of logic as a mathematical discipline is to analyze and display reasoning

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in an easily understandable fashion. While he also contributed substantially to the development of the linear notation of formal logic, he considered the later developed Existential Graphs as superior notation (see [PS00, Pei35a]).

Intuitively, the system of Existential Graphs seems equivalent to first order logic. However, a proof in the strict mathematical sense cannot be given based on Peirce's work. His description of Existential Graphs is too vague to suit the requirements of contemporary mathematics.

To solve this problem, Robert Burch studied the large range of Peirce's philosophical work and presented in [Bur91] his results on attempting an algebraization of Peirce's logic system. This algebraic logic is called *Peirce's Algebraic Logic*. He uses this logic system to prove Peirce's reduction thesis, namely, that ternary relations suffice to construct arbitrary relations, but that not all relations can be constructed from unary and binary relations alone. While this thesis is not stated explicitly in Peirce's work [Pei35b], this idea appears repeatedly.

Burch's proof depends on a restriction on the constructions allowed in PAL: the juxtaposition of disjoint graphs is only allowed as last or second-last operation. While Burch proves that the expressivity is still the same, this restriction is a major difference to the original system of Existential Graphs. Removing this restriction make the PAL-system more similar to both the system of Existential Graphs and to the system of relational algebra. The equivalence of this restriction-free PAL and relational algebra has been shown in [HCP04]. The proof of Peirce's Reduction Thesis however is more complicated if we cannot rely on this restriction.

In this paper we provide the first steps toward the proof, concentrating on the special cases of a domain with only two elements and of domains with infinitely many elements. To achieve this, we define representations of the constructed relations similar to the disjunctive normal form (DNF) known from first-order propositional logic. Taking advantage of some properties the relations in the DNF have, we can then prove the reduction thesis for the two special cases.

Organization of This Paper

In the following section we provide the basic definitions used in this paper. To simplify notation in the later parts, we in particular introduce a slight generalization of *relation* in Def. 1. Together with the definition of the PAL-graph we then introduce the disjunctive normal form in Section 3. In the following sections, we prove Peirce's Reduction Thesis for infinite and two-element domains. We conclude the paper with an outlook on further research in this area.

2 Basic Definitions

Relations in the classical sense are sets of tuples, that is relations are subsets of A^n where A is an arbitrary set (in the following called *domain*) and n is a natural number, which is the arity of the relation. However, this definition leads to unnecessary complications when discussing the interpretation of the algebraically defined PAL-graphs. While the elements of a tuple are clearly ordered, the same cannot be said about the arcs and nodes of a graph. Consequently, it is difficult and leads to cumbersome notations if we force such an order onto the interpretation of the graphs.

For this reason we introduce the following generalization, we consider relations where the places of the relations are indicated by natural numbers.

Definition 1 (Relations). *Let $I \subseteq \mathbb{N}$ be finite. A I-ARY RELATION OVER A is a set $\varrho \subseteq A^I$, i. e. a set of mappings from I to A.*

While looking slightly more complicated at first, this definition is compatible with the usual one. Any n-tuple can be interpreted as a mapping from the set $\{1, \ldots, n\}$ to A. Instead of downsets of the natural numbers as domain of the mapping, we now allow arbitrary but finite subsets of $\mathbb{N}$.

If R is an I-ary relation over A and if $J \supseteq I$, then R can be canonically extended to a J-ary relation R' by $R' := \{f : J \to A \mid f|_I \in R\}$. In this work, we use the implicit convention that all relations are extended if needed. To provide an example, let R be an I-ary relation and S be a J-ary relation. With $R \cap S$ we denote the $I \cup J$-ary relation $R' \cap S'$, where R' is R extended to $I \cup J$ and S' is S extended to $I \cup J$.

We will use the following notations to denote the arity of a relation: usually we will append the arity as lower index to the relation name. Thus R_I denotes an I-ary relation. In Sec. 5, it is convenient to append the elements of I as lower indices to R. For example, both $R_{i,j}$ and $R_{j,i}$ are names for an $\{i, j\}$-ary relation. The elements of a relation will be noted in the usual tuple-notion with round brackets, where we use the order of the lower indices. For example, both $R_{i,j} := \{(a, b), (b, b)\}$ and $R_{j,i} := \{(b, a), (b, b)\}$ denote the same relation, namely the relation $\{f_1, f_2\}$ with $f_1(i) = a$, $f_1(j) = b$ and $f_2(i) = b$, $f_2(j) = b$.

Note that $\emptyset$-ary relations are allowed. There are exactly two $\emptyset$-ary relation, namely $\emptyset$ and $\{\emptyset\}$.

From given relations, we can construct new relations. In mathematics, this usually refers to relational algebra. In this paper, we use the PAL-operations as introduced by Burch in [Bur91]. While the operations from relational algebra provide the same expressive power (see [HCP04]), the PAL operations concentrate on a different aspect. The *teridentity* is the three-place equality, that is (in the notation of standard mathematical relations as used in [HCP04]) the relation $\doteq_3 := \{(a, a, a) \mid a \in A\}$. It plays a crucial role for Peirce and also in Burch's book. The core of the Peircean Reduction Thesis is that with the teridentity any relation can be constructed from the unary and binary (or the ternary) relations, but from unary and binary relations alone one cannot construct the teridentity. This means that the teridentity would be somehow hidden in the operations from relational algebra. As the operations have the same expressivity, we can define each operation of one system by operations from the other. The operations of relational algebra can easily be expressed in PAL using the teridentity, but this is at least difficult for the identification of the first two coordinates ([HCP04], Def. 2,R3), that is $\Delta(\varrho) := \{(a_1, \ldots, a_{m-1}) \mid (a_1, a_1, \ldots, a_{m-1}) \in \varrho\}$, and for the union of relations without teridentity. Proving the Peircean Reduction Thesis will show that this is not only difficult but impossible.

In relational algebra, we can construct the teridentity in relational algebra using product, cyclic shift ζ (the tuples are rotated: see [HCP04], Def. 2,R2a) and identification of the first two coordinates from the binary identity: $\doteq_3 = \Delta(\zeta(= \times =))$. As product and cyclic shift are also in PAL, we could deduce after a final proof, that teridentity is indeed involved in the identification of the first two coordinates.

The PAL-operations found by Burch also have an easy graphical interpretation as shown in [HCP04]. We will use this notatition (see Def. 3).

1. **Negation:** If R is an I-ary relation, then
$$\neg R := A^I \backslash R$$
2. **Product:** If R is an I-ary relation, S is an J-ary relation, and we have $I \cap J = \emptyset$, then
$$R \times S := \{f : I \cup J \to A \mid (f|_I \in R) \wedge (f|_J \in S)\}$$
3. **Join:** If R is an I-ary relation with $i, j \in I$, $i \neq j$, then
$$\delta^{i,j}(R) := \{f : I \backslash \{i,j\} \to A \mid \exists F \in R : (F|_{I \backslash \{i,j\}} = f) \wedge (F(i) = F(j))\}$$

We need two further technical operations which do not belong to PAL (but they can be constructed within PAL), but which are needed in the ongoing proofs:

1. **Projection:** Let $I := \{i, j\}$ and R be an I-ary relation. Then
$$\pi_i(R) = \{(f(i)) \mid f \in R\} \quad \text{and} \quad \pi_j(R) = \{(f(j)) \mid f \in R\}$$
2. **Renaming:** If R is an I-ary relation with $i \in I$ and $j \notin I$, we set
$$\sigma^{i \to j}(R) := \{f|_{I \backslash \{i\}} \cup \{(j, f(i))\} \mid f \in R\}$$

Finally, for a given domain A, we need names for some special relations. With $\doteq_I$ we denote the I-ary identity relation, i.e. $\{f : I \to A \mid \exists a \in A \forall i \in I : f(i) = a\}$. For three-element sets I, this identity is called the I-ary TERIDENTITY. We will write $\doteq_I$ to emphasize this. With $\not\doteq_I$ we denote the complement of the teridentity. With A_I or A_I^n (we assume $|I| = n$) we denote the I-ary universal relation A^I.

After the neccessary definitions for relations, we can now define PAL-graphs over I. They are basically mathematical graphs (multi-hypergraphs), enriched with an additional structure describing the cuts. The vertices are either labelled with an element of I (then such a vertex is a free place of the graph), or with an additional sign '$*$' (in this case, the vertex denotes an unqualified, existentially quantifed object).

Definition 2 (PAL-Graphs). *For* $I \subseteq \mathbb{N}$*, a structure* $(V, E, \nu, \top, Cut, area, \kappa, \varrho)$ *is called an* I-ARY PAL-GRAPH OVER A *iff*

1. *V, E and Cut are pairwise disjoint, finite sets whose elements are called* VERTICES*,* EDGES *and* CUTS*, respectively,*

2. *$\nu : E \to \bigcup_{k\in\mathbb{N}} V^k$ is a mapping,*[1]
3. *$\top$ is a single element with $\top \notin V \cup E \cup Cut$, called the* SHEET OF ASSERTION,
4. *$area : Cut \cup \{\top\} \to \mathfrak{P}(V \cup E \cup Cut)$ is a mapping such that*
   a) *$c_1 \neq c_2 \Rightarrow area(c_1) \cap area(c_2) = \emptyset$,*
   b) *$V \cup E \cup Cut = \bigcup_{d\in Cut\cup\{\top\}} area(d)$,*
   c) *$c \notin area^n(c)$ for each $c \in Cut \cup \{\top\}$ and $n \in \mathbb{N}$ (with $area^0(c) := \{c\}$ and $area^{n+1}(c) := \bigcup\{area(d) \mid d \in area^n(c)\}$).*
5. *$\kappa : E \to \bigcup_{n\in\mathbb{N}} \mathfrak{P}(A^n)$ is a mapping with $\kappa(e) \subseteq A^n$ for $|e| = n$ (see below for the notion of $|e|$),*
6. *$\varrho : V \to I \cup \{*\}$ is a mapping such that for each $i \in I$, there is exactly one vertex v_i with $\varrho(v_i) = i$, this vertex is incident with exactly one edge and we have $v_i \in area(\top)$, and*
7. *$\mathfrak{G}$ has* DOMINATING NODES, *i.e., for each edge $e = (v_1, \ldots, v_k)$ and each incident vertex $v_i \in \{v_1, , \ldots, v_k\}$, there is $e \in area^n(cut(v_i))$ for an $n \geq 1$ (see below for the notions of $e = (v_1, \ldots, v_k)$ and $cut(v_i)$).*

For an edge $e \in E$ with $\nu(e) = (v_1, \ldots, v_k)$ we set $|e| := k$ and $\nu(e)\big|_i := v_i$. Sometimes, we also write $e\big|_i$ instead of $\nu(e)\big|_i$, and $e = (v_1, \ldots, v_k)$ instead of $\nu(e) = (v_1, \ldots, v_k)$. We set $E^{(k)} := \{e \in E \mid |e| = k\}$.

As for every $x \in V \cup E \cup Cut$ there is exactly one context $c \in Cut \cup \{\top\}$ with $x \in area(c)$, we can write $c = area^{-1}(x)$ for every $x \in area(c)$, or even more simple and suggestive: $c = cut(x)$.

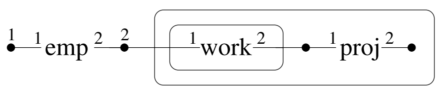
We set $V^* := \{v \in V \mid \varrho(v) = *\}$ and $V^? := \{v \in V \mid \varrho(v) \in \mathbb{N}\}$, and we set $FP(\mathfrak{G}) := I$ ('FP' stands for 'free places').

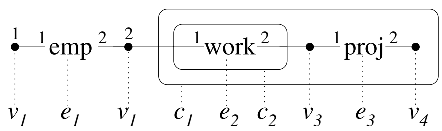
In the following, PAL-graphs will be abbreviated by PG.

An example for this definition is the following PG:

$$\begin{aligned}\mathfrak{G} := (\ &\{v_1, v_2, v_3, v_4\},\ \{e_1, e_2, e_3\},\ \{(e_1, (v_1, v_2)), (e_2, (v_2, v_3)), (e_3, (v_3, v_4))\},\\ &\top,\ \{c_1, c_2\},\ \{(\top, \{v_1, v_2, e_1, c_1\}), (c_1, \{v_3, v_4, e_3, c_2\}), (c_2, \{e_2\})\},\\ &\{(e_1, \mathrm{emp}), (e_2, \mathrm{work}), (e_3, \mathrm{proj})\},\ \{(v_1, 1), (v_2, 2), (v_3, *), (v_4, *)\}\ )\end{aligned}$$

Below, the left diagram is a possible representation of $\mathfrak{G}$. In the right diagram, we have sketched furthermore assignments of the elements (the vertices, edges, and cuts) of the $\mathfrak{G}$ to the graphical elements of the diagram. The precise conventions on how the graphs are diagrammatically represented will be given in Def. 3.





(This is a standard example for querying relational databases. If emp relates names of employees and their ids, proj relates description of projects and their ids, and work is a relation between employee ids and project describing which

[1] We set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

employee works in which project, then this graph retrieves all employees who work in all projects.)

A PG $\mathfrak{G}$ with $\mathrm{FP}(\mathfrak{G}) = I$ describes the I-ary relation of all tuples $(a_1, \ldots, a_n)$ such that when the free places of $\mathrm{FP}(\mathfrak{G})$ are replaced by $a_1, \ldots, a_n$, we obtain a graph which evaluates to true. Following the approach of [Dau03] and [Dau04], PGs have been defined in one step, and the evaluation of graphs could be defined analogously to the evaluation of concept/query graphs with cuts, which is done over the tree of contexts $Cut \cup \{\top\}$. In this paper, we follow a different approach.

PGs can be defined inductively as well, such that the inductive construction of PGs corresponds to the operations on relations. In the following, this inductive construction of PGs is introduced, and we define the semantics of the graphs along their inductive construction. Moreover, a graphical representation of PGs is provided as well.

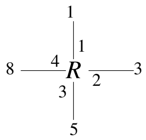
Definition 3 (Inductive Definition of PGs, Semantics, Graphical Representation).

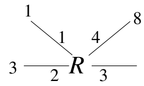
1. ***Atomar graphs:*** *Let R be an I-ary relation with $I = \{i_1, \ldots, i_n\} \neq \emptyset$. Let $R' := \{(f(i_1), \ldots, f(i_n)) \mid f \in R\}$ be the corresponding 'ordinary' n-ary relation over the domain A. The graph*

   $$(\{v_1, \ldots, v_n\}, \{e\}, \{(e, (v_1, \ldots, v_n))\}, \top, \emptyset, \emptyset, \{(e, R')\}, \{(v_1, i_1), \ldots, (v_n, i_n)\})$$

   is the atomic PG corresponding to R. If this graph is named $\mathfrak{G}$, we see that $\mathfrak{G}$ is an I-ary PG. We set $\mathcal{R}(\mathfrak{G}) := R$.

   Graphically, a vertex v of $\mathfrak{G}$ with $\varrho(v) = *$ is depicted as bold spot $\bullet$, and a vertex v with $\varrho(v) = i$ is labelled with i. The edge $e = (v_1, \ldots, v_n)$ is depicted by its label $R := \kappa(e)$, which is linked for each vertex v_i, $i = 1, \ldots, n$ to its representing sign. This line is labelled with i. For example, the following diagrams depict the same $\{1, 3, 5, 8\}$-ary relation R:

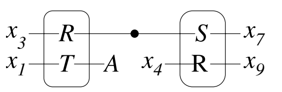
   

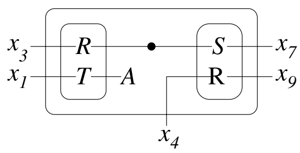
   and
   

   .

2. ***Cut Enclosure:*** *Let $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$ be an I-ary PAL-graph. Let c be a fresh cut (i.e., $c \notin E \cup V \cup Cut \cup \{\top\}$). Then let $\neg\mathfrak{G}$ be the PG defined by $(V, E, \nu, \top, Cut', area', \kappa, \varrho)$ with $Cut' := Cut \cup \{c\}$, $area'(d) := area(d)$ for $d \neq c$ and $d \neq \top$, $area'(\top) := V^?$ and $area'(c) := area(\top) \backslash V^?$. This graph is an I-ary PG. We set $\mathcal{R}(\neg\mathfrak{G}) := (R)^c := A^I \backslash \mathcal{R}(\mathfrak{G})$.*

   In the graphical notation, all elements of the graph, except the vertices labelled with a free place, are enclosed by a finely drawn, closed line, the cut-line of c. For example,

   from
   

   we obtain
   

   .

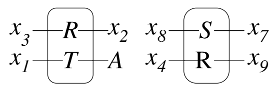
3. ***Juxtaposition:*** *Let* $\mathfrak{G}_1 := (V_1, E_1, \nu_1, \top_1, Cut_1, area_1, \kappa_1, \varrho_1)$ *be an* I*-ary PG and let* $\mathfrak{G}_2 := (V_2, E_2, \nu_2, \top_2, Cut_2, area_2, \kappa_2, \varrho_2)$ *be a* J*-ary PG such that* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *are disjoint, and* I *and* J *are disjoint. The* JUXTAPOSITION OF $\mathfrak{G}_1$ AND $\mathfrak{G}_2$ *is defined to be the PG* $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$*:*

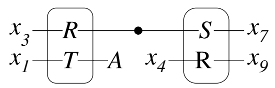
$$\mathfrak{G}_1\ \ \mathfrak{G}_2 := (V_1 \cup V_2, E_1 \cup E_2, \nu_1 \cup \nu_2, \top, Cut_1 \cup Cut_2, area, \kappa_1 \cup \kappa_2, \varrho_1 \cup \varrho_2)$$

where $\top$ *is a fresh sheet of assertion (part.* $\top \neq \top_1, \top 2$*), and we set* $area(c) := area_i(c)$ *for* $c \in Cut_i$*,* $i = 1, 2$*, and* $area(\top) := area_1(\top_1) \cup area_2(\top_2)$*. This graph is an* $I \cup J$*-ary PG. We set* $\mathcal{R}(\mathfrak{G}_1\ \ \mathfrak{G}_2) := \mathcal{R}(\mathfrak{G}_1) \times \mathcal{R}(\mathfrak{G}_2)$*.*
In the graphical notation, the juxtaposition of $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is simply noted by writing the graphs next to each other, i.e. we write:* $\mathfrak{G}_1\ \mathfrak{G}_2$*.*
4. ***Join:*** *Let* $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$ *be an* I*-ary PG, and let* $i, j \in I$ *with* $i \neq j$*. Let* v_i, v_j *be the vertices with* $\varrho(v_i) = i$ *and* $\varrho(v_j) = j$*. Let* v *be a fresh vertex. Then the* JOIN OF i AND j FROM $\mathfrak{G}$ *is*

$$\delta^{i,j}(\mathfrak{G}) := (V', E, \nu', \top, Cut, area', \kappa, \varrho')$$

with $V' := V \backslash \{v_i, v_i\} \cup \{v\}$*,* ν' *satisfies* $\nu'(e)|_k := \nu(e)|_k$ *for* $\nu(e)|_k \neq v_i, v_j$ *and* $\nu'(e)|_k := v$ *otherwise,* $area'(c) := area(c)$ *for* $c \in Cut$ *and* $area'(\top) := area(\top) \backslash \{v_i, v_i\} \cup \{v\}$*, and* $\varrho'(w) := \varrho(w)$ *for* $w \neq v$ *and* $\varrho'(v) := *$*. This graph is an* $I \backslash \{i, j\}$*-ary PG. We set* $\mathcal{R}(\delta^{i,j}(\mathfrak{G})) := \delta^{i,j}(\mathcal{R}(\mathfrak{G}))$*.*
In the graphical notation, the vertices v_i, v_j *are both replaced by the same, heavily drawn dot, which stands for an existential quantified object. For example, with joining the vertices with* 2 *and* 8*,*

from 

we obtain 

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We have seen in the definition that all inductively constructed graphs are PGs. On the other hand, for a given PG $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$, it can easily be shown by induction over the tree of contexts $Cut \cup \{\top\}$ that $\mathfrak{G}$ can be constructed with the above PAL-operations, and that different inductive constructions of $\mathfrak{G}$ yield the same semantics and the same graphical representation. Thus for each PG $\mathfrak{G}$, we have a well-defined meaning $\mathcal{R}(\mathfrak{G})$ and a well-defined graphical representation of $\mathfrak{G}$.

Graphs similar to PGs have already been studied by one of the authors in [Dau03] and [Dau04]. In [Dau03], *concept graphs with cuts*, which are based on Peirce's Existential Graphs and which, roughly speaking, correspond to closed formulas of first order logic, have been investigated. In [Dau04], concept graphs with cuts are syntactically extended to *query graphs with cuts* by adding labelled query markers to their alphabet, so query graphs with cuts are evaluated to *relations* in models. Both [Dau03] and [Dau04] focus on providing sound and complete calculi for the systems. This is done as common in mathematical logic, that is, graphs are defined as purely syntactical structures, built over an alphabet of names, which gain their meaning when their alphabet are interpreted in models.

Both query graphs with cuts and PGs are graphs which describe relations. The main difference between these types of graphs is as follows: PGs are *semantical* structures, that is, we directly assign relations to the edges of PGs, instead of assigning relation names, which then would have to be interpreted in models. Moreover, in query graphs with cuts, object names may appear, objects are classified by *types*, and we have orders on the set of types and relation names. From this point of view, PGs can be considered to be restrictions of query graphs with cuts, but this restriction is only a minor one.

3 Disjunctive Normal Form for PGs

Let $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$ be a PG. Let $\sim$ be the smallest equivalence relation on V such that for all $e = (v_1, \ldots, v_n)$, there is $v_1 \sim v_2 \sim \ldots \sim v_n$, and for $v \sim v'$, we say that v and v' are CONNECTED. As for each free place $i \in \mathrm{FP}(\mathfrak{G})$ there exists a uniquely given vertex $w_i \in V$ with $\varrho(w_i) = i$, this equivalence relation is transferred to $\mathrm{FP}(\mathfrak{G})$ by setting $i \sim j :\Leftrightarrow w_i \sim w_j$. Finally we set

$$P(\mathfrak{G}) := \{[i]_\sim \mid i \in \mathrm{FP}(\mathfrak{G})\} \cup \{\emptyset\} \quad .$$

$P(\mathfrak{G})$ is simply the set of all equivalence classes, together with the empty set $\emptyset$. Next we show that for a PG $\mathfrak{G}$, the relation $\mathcal{R}(\mathfrak{G})$ can be described as a union of intersections of I-ary relations with $I \in P(\mathfrak{G})$. In the proof, we may obtain $\emptyset$-ary relations , that is why we have to add $\emptyset$ to $P(\mathfrak{G})$.

Theorem 1 (Disjunctive Normal Form (DNF) for Relations described by PGs). *Let $\mathfrak{G}$ be a PG. Then there is a $n \in \mathbb{N}$, and for each $m \in \{1, \ldots, n\}$ and for each class $p \in P(\mathfrak{G})$ there is a p-ary relation R_p^m, such that we have*

$$\mathcal{R}(\mathfrak{G}) = \bigcup_{m \in \{1,\ldots,n\}} \bigcap_{p \in P(\mathfrak{G})} R_p^m$$

The relations R_p^m shall be called GROUND RELATIONS OF $\mathfrak{G}$.

Proof: The proof is done by induction over the construction of PGs.

1. Atomar graphs: If R is an relation and $\mathfrak{G}_R$ be the corresponding atomar graph, it is easy to see that the theorem holds for $\mathfrak{G}_R$ by setting $n := 1$ and $R_p^1 := R$.
2. Juxtaposition: Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two PGs with $\mathbb{N}(\mathfrak{G}_1) \cap \mathbb{N}(\mathfrak{G}_2) = \emptyset$. If we use the letter R to denote the relations of $\mathfrak{G}_1$ and the letter S to denote the relations of $\mathfrak{G}_2$, we have

$$\mathcal{R}(\mathfrak{G}_1) = \bigcup_{m \in \{1,\ldots,n_1\}} \bigcap_{p \in P(\mathfrak{G}_1)} R_p^m \quad \text{and} \quad \mathcal{R}(\mathfrak{G}_2) = \bigcup_{m \in \{1,\ldots,n_2\}} \bigcap_{p \in P(\mathfrak{G}_2)} S_p^m$$

Thus we have with the canonical extension of the ground relations

$$\mathcal{R}(\mathfrak{G}) = \left(\bigcup_{m \in \{1,\ldots,n_1\}} \bigcap_{p \in P(\mathfrak{G}_1)} R_p^m \right) \quad \cap \quad \left(\bigcup_{m \in \{1,\ldots,n_2\}} \bigcap_{p \in P(\mathfrak{G}_2)} S_p^m \right)$$

Now an application of the distributive law, using $P(\mathfrak{G}) = P(\mathfrak{G}_1) \cup P(\mathfrak{G}_2)$ and $n := n_1 + n_2$, yields the theorem for $\mathfrak{G}$.

3. Cut enclosure: We consider $\neg\mathfrak{G}$. Due to the induction hypothesis, we have

$$\mathcal{R}(\mathfrak{G}) = \bigcup_{m \in \{1,\ldots,n\}} \bigcap_{p \in P(\mathfrak{G})} R_p^m$$

Thus, using De Morgan's law, we have

$$\mathcal{R}(\neg\mathfrak{G}) = (\bigcup_{m \in \{1,\ldots,n\}} \bigcap_{p \in P(\mathfrak{G})} R_p^m)^c = \bigcap_{m \in \{1,\ldots,n\}} \bigcup_{p \in P(\mathfrak{G})} (R_p^m)^c$$

Similar to the last case, we apply the distributive law to obtain a union of intersections of relations. Due to the distributive law, given a class $p \in P(\mathfrak{G})$, the p-ary ground relations of $\neg\mathfrak{G}$ are intersections of 0 up to d relations $(R_p^m)^c$, and these intersections are relations over p, too. Thus the theorem holds for $\neg\mathfrak{G}$ as well.

4. Join: We consider $\mathfrak{G}$ and two distinct free places $i, j \in \mathbb{N}(\mathfrak{G})$. With $q := ([i]_\sim \cup [j]_\sim)\backslash\{i,j\}$, we have $P(\delta^{i,j}(\mathfrak{G})) = P(\mathfrak{G})\backslash\{[i]_\sim, [j]_\sim\} \cup \{q\}$. Now we conclude

$$\begin{aligned}\delta^{j,k}(\mathcal{R}(\mathfrak{G})) &= \delta^{j,k}\left(\bigcup_{m \in \{1,\ldots,n\}} \bigcap_{p \in P(\mathfrak{G})} R_p^m\right)\\ &= \bigcup_{m \in \{1,\ldots,n\}} \delta^{j,k}\left(\bigcap_{p \in P(\mathfrak{G})} R_p^m\right)\\ &= \bigcup_{m \in \{1,\ldots,n\}} \bigcap_{p \in P(\mathfrak{G}),\, j,k \notin p} \left(R_p^m \cap \delta^{j,k}\left(R^e_{[i]_\sim} \cap R^e_{[j]_\sim}\right)\right)\end{aligned}$$

As $\delta^{j,k}(R^e_{[i]_\sim} \cap R^e_{[j]_\sim})$ is a q-ary relation, we are done. $\Box$

4 Proof of the Peircean Reduction Thesis for Infinite Domains

Using the theorem from the last section, the first instance of the Peircean Reduction Thesis can easily be shown as a corollary. Before that, some observations about the theorem and its proof are provided.

For a given PG $\mathfrak{G} := (V, E, \nu, \top, Cut, area, \kappa, \varrho)$, the relations R_p^m in Thm. 1 depend on the relations which appear in $\mathfrak{G}$, i.e., they depend on κ, but the proof of Thm. 1 yields that the number n of disjuncts $\bigcap_{p \in P(\mathfrak{G})} R_p^m$ does not depend on κ. That is, if we denote n by $n(\mathfrak{G})$, two PGs $\mathfrak{G}_1$, $\mathfrak{G}_2$ which differ only in κ, i.e., $\mathfrak{G}_1 = (V, E, \nu, \top, Cut, area, \kappa_1, \varrho)$ and $\mathfrak{G}_2 = (V, E, \nu, \top, Cut, area, \kappa_2, \varrho))$, satisfy $n(\mathfrak{G}_1) = n(\mathfrak{G}_2)$.

Now we are prepared to prove the reduction thesis for infinite domains with a simple counting argument.

Corollary 1 (Reduction Thesis for infinite Domains). *Let $\mathfrak{G}$ be an I-ary PG over a domain A with $|I| = 3$, and let each relation in $\mathfrak{G}$ have an arity ≤ 2. If we have $|A| > n(\mathfrak{G})$, then $\mathcal{R}(\mathfrak{G}) \neq \doteq_I$. Particularly, for an infinite set A, there exists no PG which evaluates to the teridentity on A.*

Proof: W.l.o.g. let $\mathrm{FP}(\mathfrak{G}) = \{1, 2, 3\}$. As each relation of $\mathfrak{G}$ has an arity ≤ 2, we cannot have $1 \sim 2 \sim 3$. For the proof, we assume that we have two equivalent free places (the case $P(\mathfrak{G}) = \{\{1\}, \{2\}, \{3\}, \emptyset\}$ can be proven analogously), and w.l.o.g. let $2 \sim 3$. Now Thm. 1 yields

$$\mathcal{R}(\mathfrak{G}) = \bigcup_{m \in \{1,\ldots,n\}} R^m_\emptyset \cap R^m_1 \cap R^m_{2,3}$$

Now let A be a domain with $|A| > n(\mathfrak{G})$. Assume $\mathcal{R}(\mathfrak{G}) = \doteq_{1,2,3}$. Then there exists an $m \leq n$ and distinct $a, b \in A$ with $(a, a, a), (b, b, b) \in R^m_\emptyset \cap R^m_1 \cap R^m_{2,3}$. We obtain $R^m_\emptyset = \{\emptyset\}$, $(a), (b) \in R^m_1$ and $(a, a), (b, b) \in R^m_{2,3}$, thus we have $(a, b, b), (b, a, a) \in R^m_\emptyset \cap R^m_1 \cap R^m_{2,3}$, too, which is a contradiction. □

5 Peirce's Reduction Thesis for Two-Element Domains

In the last section, we have proven Peirce's reduction thesis with a counting argument. But this argument does not apply to finite domains. For example, if $A = \{a_1, \ldots, a_n\}$ is an n-element domain, one might think that we can construct a PG such that its DNF has n disjuncts, each of them evaluating to exactly one triple $\{(a_i, a_i, a_i)\}$. In this section, we show that for two-element domains $A = \{a, b\}$, there is no PG $\mathfrak{G}$ with $\mathcal{R}(\mathfrak{G}) = \doteq_3$. This is done by classifying the relations over A into classes such that no class is suited to describe (a, a, a) in one disjunct and (b, b, b) in another disjunct, and by proving that the operations on relations 'respect' the classes.

For a relation $R_{i,j}$, we set $\gamma^i_x(R_{i,j}) := \{y \mid (x, y) \in R_{i,j}\}$. Now we define the following classes:[2]

$$\begin{array}{lll}
C^{i,j}_a := \{R_{i,j} \mid \gamma^i_a(R_{i,j}) \supseteq \gamma^i_b(R_{i,j})\} & \text{and} & C^i_a := \{\emptyset, \{a\}, \{a, b\}\} \\
C^{i,j}_b := \{R_{i,j} \mid \gamma^i_b(R_{i,j}) \supseteq \gamma^i_a(R_{i,j})\} & \text{and} & C^i_b := \{\emptyset, \{b\}, \{a, b\}\} \\
C^{i,j}_{\doteq} := \{\emptyset^2_{i,j}, \doteq_{i,j}, A^2_{i,j}\}\ ,\ C^{i,j}_{\not\doteq} := \{\emptyset^2_{i,j}, \not\doteq_{i,j}, A^2_{i,j}\} & \text{and} & C^i_{a,b} := \{\{a, b\}\}
\end{array}$$

For our purpose, the intuition behind this definition is as follows: $R_{i,j} \in C^{i,j}_a$ means that b cannot be separated (in position i) resp. $R_{i,j} \in C^{i,j}_b$ means that a cannot be separated (in position i).

PGs are built up inductively with the construction steps juxtaposition, cut enclosure, and join. The next three lemmata show how the classes are respected by the correponding operations for relations.

[2] Recall the notion $R_{i,j}$ for an $\{i, j\}$-ary relation, and recall that both $R_{i,j}$ and $R_{j,i}$ denote the same relation. But for the definition of γ and the classes $C^{i,j}_a$, $C^{i,j}_b$, the order of the indices is important. For example, given a relation $R_{i,j}$ $(= R_{j,i})$, it might happen that $R_{i,j} \in C^{i,j}_a$ and $R_{j,i} \in C^{j,i}_b$.

To ease the notation, we abbreviate the composition of product and join. So let $\mathfrak{G}$ be an PG and let $i, j \in \mathrm{FP}(\mathfrak{G})$ with $i \not\sim j$. Then we write

$$R_{[i]_\sim} \circ_{j,k} R_{[j]_\sim} := \delta^{j,k}\left(R_{[i]_\sim} \cap R_{[j]_\sim}\right) \quad (= \delta^{j,k}\left(R_{[i]_\sim} \times R_{[j]_\sim}\right))$$

Next we investigate how these classes are respected by the operations on relations. We start with the classes $C_a^{i,j}$ and $C_b^{i,j}$.

Lemma 1 (Class-Inheritance for $C_a^{i,j}$ and $C_b^{i,j}$). *Let $R_{i,j} \in C_a^{i,j}$. Then:*

1. *If $S_{k,l}$ is arbitrary, then $R_{i,j} \circ_{j,k} S_{k,l} \in C_a^{i,l}$*
2. *If S_k is arbitrary, then $R_{i,j} \circ_{j,k} S_k \in C_a^i$*
3. *$\neg(R_{i,j}) \in C_b^{i,j}$*
4. *$C_a^{i,j}$ is closed under (possibly empty) finite intersections (with $\bigcap \emptyset = A_{i,j}^2$).*

The analogous propositions hold for $R_{i,j} \in C_b^{i,j}$ as well.

Proof:

1. Let $(b, y) \in R_{i,j} \circ_{j,k} S_{k,l} \in C_a^{i,l}$. Then there exists x with $(b, x) \in R_{i,j}$ and $(x, y) \in S_{k,l}$. From $R_{i,j} \in C_a^{i,j}$ we obtain $(a, x) \in R_{i,j}$, thus we have $(a, y) \in R_{i,j} \circ_{j,k} S_{k,l}$ as well. So we conclude $R_{i,j} \circ_{j,k} S_{k,l} \in C_a^{i,l}$.
2. Done analogously to the last case.
3. We have $\gamma_x(\neg R_{i,j}) = (\gamma_x(R_{i,j}))^c$ for $x \in \{a, b\}$. So we get $R_{i,j} \in C_a^{i,j} \Leftrightarrow \gamma_a(R_{i,j}) \supseteq \gamma_b(R_{i,j}) \Leftrightarrow (\gamma_a(R_{i,j}))^c \subseteq (\gamma_b(R_{i,j}))^c \Leftrightarrow \gamma_a(\neg R_{i,j}) \subseteq \gamma_b(\neg R_{i,j}) \Leftrightarrow \neg R_{i,j} \in C_b^{i,j}$
4. If $R_{i,j}^n, n \in \mathbb{N}$ are arbitrary relations, we have $\gamma_x(\bigcap_{n\in N} R_{i,j}^n) = \bigcap_{n \in N} \gamma_x(R_{i,j}^n)$, which immediately yields this proposition. $\square$

The next lemma corresponds to Lem. 1, now for the class $C_{\doteq}^{i,j}$.

Lemma 2 (Class-Inheritance for $C_{\doteq}^{i,j}$). *Let $R_{i,j} \in C_{\doteq}^{i,j}$. Then:*

1. *If $S_{k,l} \in C_{\doteq}^{k,l}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C_{\doteq}^{i,l}$.*
   If $S_{k,l} \in C_{\not\doteq}^{k,l}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C_{\not\doteq}^{i,l}$.
   If $S_{k,l} \in C_a^{k,l}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C_a^{i,l}$.
   If $S_{k,l} \in C_b^{k,l}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C_b^{i,l}$.
2. *If $S_k \in C_{a,b}^k$, then $R_{i,j} \circ_{j,k} S_k \in C_{a,b}^i$.*
   If $S_k \in C_a^k$, then $R_{i,j} \circ_{j,k} S_k \in C_a^i$.
   If $S_k \in C_b^k$, then $R_{i,j} \circ_{j,k} S_k \in C_b^i$.
3. *$\neg(R_{i,j}) \in C_{\not\doteq}^{i,l}$.*
4. *$C_{\doteq}^{i,j}$ is closed under (possibly empty) finite intersections.*

Proof:

1. For each relation $R_{k,l}$ we have $\doteq_{i,j} \circ_{j,k} R_{k,l} = \sigma^{k \to i}(R_{k,l})$.
   For each relation $R_{k,l}$ we have $A_{i,j}^2 \circ_{j,k} R_{k,l} = A_i \times \pi_l(R_{k,l})$. Particulary, for each relation $R_{i,j}$, we have both $A_{i,j}^2 \circ_{j,k} R_{k,l} \in C_a^{i,l}$ and $A_{i,j}^2 \circ_{j,k} R_{i,j} \in C_b^{i,l}$. Moreover, for $R_{k,l} \in C_{\doteq}^{k,l}$ or $R_{k,l} \in C_{\not\doteq}^{k,l}$, we have $A_{i,j}^2 \circ_{j,k} R_{k,l} = A_{i,l}^2$.
   From these obervations we conclude this proposition.

2. For each relation R_k we have $\doteq_{i,j} \circ_{j,k} R_k = \sigma^{k \to i}(R_k)$.
   For each relation $R_k \neq \emptyset$ we have $A^2_{i,j} \circ_{j,k} R_k = A_i$, for $R_k = \emptyset$ we have $A^2_{i,j} \circ_{j,k} R_k = \emptyset$.
   From these obervations we conclude this proposition.
3. Trivial.
4. Trivial. □

Of course, we have an analogous lemma for the class $C^{i,j}_{\neq}$. The proof is analogous to the last proof and henceforth omitted.

Lemma 3 (Class-Inheritance for $C^{i,j}_{\neq}$).
Let $R_{i,j} \in C^{i,j}_{\neq}$. Then we have:

1. *If $S_{k,l} \in C^{k,l}_{\doteq}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C^{i,l}_{\doteq}$.*
   If $S_{k,l} \in C^{k,l}_{\neq}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C^{i,l}_{\neq}$.
   If $S_{k,l} \in C^{k,l}_{a}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C^{i,l}_{a}$.
   If $S_{k,l} \in C^{k,l}_{b}$, then $R_{i,j} \circ_{j,k} S_{k,l} \in C^{i,l}_{b}$.
2. *If $S_k \in C^k_{a,b}$, then $R_{i,j} \circ_{j,k} S_k \in C^i_{a,b}$.*
   If $S_k \in C^k_{a}$, then $R_{i,j} \circ_{j,k} S_k \in C^i_{b}$.
   If $S_k \in C^k_{b}$, then $R_{i,j} \circ_{j,k} S_k \in C^i_{a}$.
3. *$\neg(R_{i,j}) \in C^{i,l}_{\doteq}$.*
4. *$C^{i,j}_{\neq}$ is closed under (possibly empty) finite intersections.*

Theorem 2 (Properties of the relations in the DNF for PGs). *Let $\mathfrak{G}$ a PG. Let $i \in FP(\mathfrak{G})$ with $\{i\} \in P(\mathfrak{G})$.*
Then one of the following properties holds:

1. *$R^m_i \in C^i_a$ for all $m \in \{1, \ldots, n\}$*
2. *$R^m_i \in C^i_b$ for all $m \in \{1, \ldots, n\}$*
3. *$R^m_i \in C^i_{a,b}$ for all $m \in \{1, \ldots, n\}$*

Let $i, j \in FP(\mathfrak{G})$ with $i \sim j$. Then one of the following properties holds:

1. *$R^m_{i,j} \in C^{i,j}_{\doteq}$ for all $m \in \{1, \ldots, n\}$*
2. *$R^m_{i,j} \in C^{i,j}_{\neq}$ for all $m \in \{1, \ldots, n\}$*
3. *$R^m_{i,j} \in C^{i,j}_{a}$ for all $m \in \{1, \ldots, n\}$*
4. *$R^m_{i,j} \in C^{i,j}_{b}$ for all $m \in \{1, \ldots, n\}$*

Proof: The proof is done by induction over the construction of PAL-graphs.

Atomar graphs: For each relation $R_{i,j}$ we have $R_{i,j} \in C^{i,j}_a \cup C^{i,j}_b \cup \{\doteq, \neq\}$. Thus it is easy to see that the theorem holds for atomar graphs.

Juxtaposition: If we consider the juxtaposition of two graphs $\mathfrak{G}_1$, $\mathfrak{G}_2$, then the ground relations of the juxtaposition are the ground relations of $\mathfrak{G}_1$ and the ground relations of $\mathfrak{G}_2$.

Cut enclosure: As said in the proof of Thm.1, given a class $p \in P(\mathfrak{G})$, the p-ary ground relations of $\neg\mathfrak{G}$ are intersections of 0 up to d relations $(R^m_p)^c$, where the relations R_p^m are the p-ary ground relations of $\mathfrak{G}$. First of all, due to Lem. 1.3., 2.3., 3.3., the set of all complements of the ground relations fulfill the property of this theorem. Moreover, due to Lem. 1.4., 2.4., 3.4., all classes $C_{\doteq}^{i,j}$, $C_{\neq}^{i,j}$, $C_a^{i,j}$, $C_b^{i,j}$ are closed under (possibly empty) intersections. Thus the theorem holds for $\neg\mathfrak{G}$ as well.

Join: We consider $\delta^{j,k}(\mathfrak{G})$. We have $\mathbb{N}(\delta^{j,k}(\mathfrak{G})) = \mathrm{FP}(\mathfrak{G})\backslash\{j,k\}$. Due to the proof of Thm. 1, we have to show that the proposition holds for the new ground relations $R_{[j]_\sim}^m \circ_{j,k} R_{[k]_\sim}^m = \delta^{j,k}(R_{[j]_\sim}^m \cap R_{[k]_\sim}^m)$.

First we consider the case that $\{j\},\{k\} \in P(\mathfrak{G})$. We have $P(\delta^{j,k}(\mathfrak{G})) = P(\mathfrak{G})\backslash\{\{j\},\{k\}\}$. For $p \neq \{j\},\{k\}$, the ground relations of $\mathfrak{G}$ and of $\delta^{j,k}(\mathfrak{G})$ which are not over j or k (or over $\emptyset$) are the same, thus we are done. The case $j \sim k$, i.e. $\{j,k\} \in P(\mathfrak{G})$, can be handled analogously.

Next we consider the case that there is an i with $i \sim j$, but there is no l with $k \sim l$. We have $P(\delta^{j,k}(\mathfrak{G})) = P(\mathfrak{G})\backslash\{\{i,j\},\{k\}\} \cup \{\{i\}\}$. The new ground relations are of the form $R_{i,j}^m \circ_{j,k} R_k^m$. We have to do a case distinction, both for $R_{i,j}^m$ and R_k^m.

Assume for example $R_{k,l}^m \in C_{\doteq}^{i,j}$ for all $m \leq n$ and $R_k^m \in C_{a,b}^{i,j}$, then $R_{i,j} \circ_{j,k} R_k \in C_{a,b}^{i,j}$ due to Lem. 2.2. All other cases are proven analogously with Lem. 1.2., 2.2., 3.2..

The case when there is an l with $k \sim l$, but there is no i with $i \sim j$ can be done analogously to the last case (now looking which properties $R_{l,k}^m$ has. Note that we have to consider $R_{l,k}^m$ instead of $R_{k,l}^m$).

Now we finally consider the case that there are i,k with $i \sim j$ and $k \sim l$. Then $P(\delta^{j,k}(\mathfrak{G})) = P(\mathfrak{G})\backslash\{\{i,j\},\{k,l\}\} \cup \{\{i,l\}\}$. The new ground relations we obtain are $R_{i,j}^m \circ_{j,k} R_{k,l}^m$ with $m \leq n$. Again, we have to do a case distinction, both for the classes $\{i,j\}$ and $\{k,l\}$. This case dinstinction is done analogously to the last case one, now using Lem. 1.2., 2.2. and 3.2. □

Corollary 2 (Reduction Thesis for two-element Domains). *Let A be a domain with $|A| = 2$, and let $\mathfrak{G}$ be a ternary PG over A where each relation has an arity ≤ 2. Then $\mathcal{R}(\mathfrak{G}) \neq \doteq_3$.*

Like in the proof of Cor. 1, let $\mathrm{FP}(\mathfrak{G}) = \{1,2,3\}$, and let

$$\mathcal{R}(\mathfrak{G}) = \bigcup_{m \in \{1,\ldots,n\}} R_\emptyset^m \cap R_1^m \cap R_{2,3}^m$$

be a DNF for $\mathcal{R}(\mathfrak{G})$.

Assume $\mathcal{R}(\mathfrak{G}) = \doteq_{1,2,3}$. Due to $R_1^m \cap R_{2,3}^m = R_1^m \times R_{2,3}^m$, each relation R_1^m contains at most one element (a) or (b). On the other hand, there must then be an m with $R_1^m = (a)$ and an f with $R_1^f = (b)$. However, one of the three classes C_a^i, C_b^i, $C_{a,b}^i$ contains the relations R_1^m and R_1^f, but none of the three classes contains both $\{(a)\}$ and $\{(b)\}$. Contradiction. □

6 Further Research

The methods and ideas presented in this paper will be continued to a complete proof of Peirce's Reduction Thesis. The main structure will be similar to the second proof presented here, but the necessary generalizations still pose problems in some details.

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